

CONNECTIVITY OF PSEUDOMANIFOLD GRAPHS FROM AN ALGEBRAIC POINT OF VIEW

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ABSTRACT. The connectivity of graphs of simplicial and polytopal complexes is a classical subject going back at least to Steinitz, and the topic has since been studied by many authors, including Balinski, Barnette, Athanasiadis and Björner. In this note, we provide a unifying approach which allows us to obtain more general results. Moreover, we provide a relation to commutative algebra by relating connectivity problems to graded Betti numbers of the associated Stanley–Reisner rings.

1. CONNECTIVITY OF THE UNDERLYING GRAPH

Let Δ be a finite simplicial complex on the vertex set $[n] := \{1, \dots, n\}$. The **underlying graph** (or **1-skeleton**) G_Δ of Δ is the graph obtained by restricting Δ to faces of cardinality at most two.

A graph G is said to be **k -connected** if it has at least k vertices and removing any subsets of vertices of cardinality less than k results in a connected graph. The **(vertex-)connectivity** κ_G of G is the maximum number k such that G is k -connected.

The classical Steinitz’s theorem [Ste22] asserts that a graph G is the underlying graph of a 3-polytope if and only if G is 3-connected and planar. In 1961, Balinski extended the “only if” direction of Steinitz’s theorem by showing that the underlying graph of a d -polytope is d -connected, cf. [Zie95]. David Barnette showed that the same bound is also valid for the connectivity number of underlying graphs of $(d-1)$ -dimensional pseudomanifolds [Bar82].

Athanasiadis [Ath11] showed that if the pseudomanifold is also flag (i.e. the clique complex of its 1-skeleton), then this lower bound can be improved to $2d-2$. Björner and Vorwerk quantified this connection using the notion of banner complexes [BV14].

The purpose of this note is to provide a unifying approach which allows us to obtain more general results.

The proof is inspired by a relation of connectivity to the Hochster’s formula (observed in [Goo14]) from commutative algebra and simple estimates for the size of certain flag complexes [ANT14].

2. BASICS IN COMMUTATIVE ALGEBRA

We start by recalling some notions, and refer to [MS05, HH11] for exact definitions and more details. Let I be a graded ideal in the polynomial ring $S = \mathbb{k}[x_1, \dots, x_n]$ in n variables over a field \mathbb{k} . Let

$$\mathbf{F}_{S/I} := 0 \rightarrow F_p \rightarrow F_{p-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I \rightarrow 0,$$

be the minimal graded free resolution of S/I , with $F_i = \bigoplus_j S(-j)^{b_{i,j}}$ in homological degree i . The number $b_{i,j} = b_{i,j}(S/I)$ is the **graded Betti number** of S/I in **homological degree** i and **internal degree** j . The length of the j -th row in the Betti table will be denoted by $lp_j(S/I)$, that is

$$lp_j(S/I) := \max\{i \mid b_{i,i+j-1}(S/I) \neq 0\}.$$

We also denote by $t_i(S/I)$ the **maximum internal degree** of a minimal generator in the homological degree i that is $\max\{j \mid b_{i,j} \neq 0\}$. The **projective dimension** of S/I is the maximum i such that $b_{i,j} \neq 0$, for some j . The **regularity** of S/I is defined to be $\max_i \{t_i(S/I) - i\}$.

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3. CONNECTIVITY VIA GRADED BETTI NUMBERS

Let Δ be a simplicial complex on the vertex set $[n]$. The **Stanley–Reisner ideal** $I_\Delta \subset S$ of Δ is the ideal generated by monomials $\mathbf{x}_F := \prod_{i \in F} x_i$ for all F not in Δ . The quotient ring $\mathbb{k}[\Delta] = S/I_\Delta$ is called the **face ring** of Δ . In this case, **Hochster’s formula** provides an interpretation of the graded Betti numbers in terms of the reduced homology of induced sub-complexes of Δ . More precisely, it asserts that

$$b_{i,j}(\mathbb{k}[\Delta]) = \sum_{\#W=j} \dim_{\mathbb{k}} \tilde{H}_{j-i-1}(\Delta_W).$$

In [Goo14, Theorem 3.1] it was observed that the graded Betti numbers computed in Hochster’s formula are naturally connected to connectivity of the underlying graphs.

Proposition 1. *Let Δ be a simplicial complex on the vertex set $[n]$ and κ be the connectivity number of its underlying graph. Then one has*

$$\kappa + lp_2(\mathbb{k}[\Delta]) = n - 1.$$

Proof. It suffices to observe that $\kappa = n - \max\{\#W : \tilde{H}_0(X_W) \neq 0\} = n - 1 - lp_2(\mathbb{k}[\Delta])$.

Alternatively (and algebraically) it suffices to observe that Δ and $\text{Cl}(G)$, the clique complex of the underlying graph G of Δ , both have the same connectivity. Hence, it suffices to verify the result in the case of flag complexes and we may assume that $\Delta = \text{Cl}(G)$. The result follows from [Goo14, Theorem 3.1]. \square

Before presenting our next result, we shall introduce two properties.

Definition 2. Let I be a graded ideal such that S/I is of regularity r . Set $m = lp_r(S/I)$. We say S/I satisfies the **property \mathfrak{A}** if

- (1) $lp_2(S/I) \leq m$,
- (2) $b_{m-i, m-i+1}(S/I) \leq b_{i, i+r-1}(S/I)$.

We also say that S/I satisfies the **property \mathfrak{B}_s** if for all $i < s$ one has $t_i(S/I) < r + i - 1$.

Remark 3. Relations to Poincaré duality and the Koszul property

- (1) If S/I is Gorenstein, then it satisfies the property \mathfrak{A} . However, the property only requires a much simpler property than Poincaré–Lefschetz duality; a simple inequality shall be enough, see Lemma 8.
- (2) If I is generated by quadratic monomials, then it is easy to see that $t_s(S/I) \leq 2s$ for all s and therefore S/I satisfies \mathfrak{B}_{r-1} . This fact is valid more generally when S/I is Koszul as was shown by Backelin in [Bac88], see also Kempf [Kem90].

Proposition 4. *Let I be a graded ideal in polynomial ring S . Moreover, assume that S/I has regularity r and satisfies the properties \mathfrak{A} and \mathfrak{B}_s . Then one has*

$$s \leq lp_r(S/I) - lp_2(S/I).$$

Proof. We have

$$\begin{aligned} lp_r(S/I) - lp_2(S/I) &= m - \max\{j \mid b_{j,j+1}(S/I) \neq 0\} \\ &= \min\{m - j \mid b_{j,j+1}(S/I) \neq 0\} \\ &\geq \min\{m - j \mid b_{m-j, m-j+r-1}(S/I) \neq 0\} \quad (\text{Property } \mathfrak{A}) \\ &= \min\{k \mid b_{k, k+r-1}(S/I) \neq 0\} \\ &\geq \min\{k \mid t_k(S/I) \geq k + r - 1\} \end{aligned}$$

where the last term is at least s by Property \mathfrak{B}_s . \square

By Bakelin’s result the regularity of a Koszul ring is bounded above by its projective dimension. As an immediate consequence of the previous result, we get the following tight bound for Gorenstein Koszul rings.

Corollary 5. *The regularity of a Gorenstein Koszul ring S/I is at most $\text{projdim}(S/I) - lp_2(S/I) + 1$. \square*

Theorem 6. *Let Δ be a $(d - 1)$ -dimensional simplicial complex with nontrivial top-homology. Also, assume that $\mathbb{k}[\Delta]$ satisfies the properties \mathfrak{A} and \mathfrak{B}_s . Then the underlying graph is $(d + s - 1)$ -connected.*

Proof. Note that the regularity of $\mathbb{k}[\Delta]$ is equal to d since Δ has nontrivial top-homology. So, it follows from Proposition 4 that

$$s \leq lp_d(\mathbb{k}[\Delta]) - lp_2(\mathbb{k}[\Delta]).$$

Note that $lp_d(\mathbb{k}[\Delta]) = n - d$. So, by Corollary 1 we get

$$s \leq n - d - (n - \kappa_\Delta - 1),$$

where κ_Δ stands for the connectivity number of the underlying graph of Δ . Therefore

$$\kappa_\Delta \geq d + s - 1. \quad \square$$

Remark 7. As a special case of Theorem 6, we can consider Δ to be Gorenstein*. Then $\mathbb{k}[\Delta]$ satisfies the property \mathfrak{B}_1 . Moreover, if Δ is also flag, then it satisfies the property \mathfrak{B}_{d-1} , since $t_i(\mathbb{k}[\Delta]) \leq 2i$ for any i and $\mathbb{k}[\Delta]$ is d -regular.

4. A POINCARÉ–LEFSCHETZ-TYPE INEQUALITY FOR MINIMAL CYCLES

Recall that a **minimal d -cycle** Σ (w.r.t. a coefficient ring R) is a pure d -dimensional complex that supports precisely one homology d -class ζ whose support is the complex itself. For instance, every pseudomanifold is a minimal cycle (over $\mathbb{Z}/2\mathbb{Z}$); and so is every triangulation of a closed, connected manifold.

Lemma 8. *Let Σ denote any minimal d -cycle and W a subset of the vertex-set $V(\Sigma)$. Then*

$$\text{rk } \tilde{H}_0(\Sigma_W) \leq \text{rk } \tilde{H}_{d-1}(\Sigma_{V(\Sigma) \setminus W})$$

Proof. Since Σ supports a global d -cycle (by minimality), we have an injection

$$H^0(\Sigma_W) \hookrightarrow H_d(\Sigma, \Sigma \setminus \Sigma_W).$$

To see this, notice that the restriction of the global d -cycle to any connected component of Σ_W induces a relative cycle for $(\Sigma, \Sigma \setminus \Sigma_W)$.

Now, since $\Sigma_{V(\Sigma) \setminus W}$ is homotopically equivalent to $\Sigma \setminus \Sigma_W$, the exact sequence

$$0 \rightarrow \tilde{H}_d(\Sigma) \rightarrow \tilde{H}_d(\Sigma, \Sigma_{V(\Sigma) \setminus W}) \rightarrow \tilde{H}_{d-1}(\Sigma_{V(\Sigma) \setminus W}) \rightarrow \cdots,$$

implies

$$\begin{aligned} & \text{rk } \tilde{H}_{d-1}(\Sigma_{V(\Sigma) \setminus W}) + 1 \\ &= \text{rk } \tilde{H}_{d-1}(\Sigma_{V(\Sigma) \setminus W}) + \text{rk } \tilde{H}_d(\Sigma) \\ &\geq \text{rk } \tilde{H}_d(\Sigma, \Sigma_{V(\Sigma) \setminus W}) \\ &\geq \text{rk } H^0(\Sigma_W). \end{aligned} \quad \square$$

5. APPLICATIONS TO CONNECTIVITY OF MANIFOLDS

Let Δ be a $(d-1)$ -dimensional simplicial complex on the vertex set $V(\Delta)$. Recall the notion of banner complexes of [BV14]:

- A subset W of $V(\Delta)$ is called **complete** if every two vertices of W form an edge of Δ .
- A complete set $W \subseteq V(\Delta)$ is **critical** if $W \setminus \{v\}$ is a face of Δ for some $v \in W$.
- We say that Δ is **banner** if every critical complete set W of size at least d is a face of Δ .
- We define the **banner number** of Δ to be

$$b(\Delta) = \min \left\{ b : \begin{array}{l} \text{lk}_\sigma \Delta \text{ is banner or the boundary of the 2-simplex} \\ \text{for all faces } \sigma \in \Delta \text{ of cardinality } b \text{ and degree } d \end{array} \right\},$$

where the **degree** of a face is the maximal cardinality of a facet containing it.

Note that our notions of banner complexes and banner numbers are slightly more general than the ones introduced in [BV14]. However, if the complex is pure the definitions coincide.

Lemma 9. *Let Δ be a $(d-1)$ -dimensional simplicial complex.*

- (a) *If σ is a face of degree d in Δ , then $b(\text{lk}_\sigma \Delta) \leq \max\{0, b(\Delta) - \#\sigma\}$.*
- (b) *If Δ has nontrivial top-homology and $b(\Delta) < d-2$, then every induced subcomplex of Δ having nontrivial $(d-2)$ -homology has at least $2d-2-b(\Delta)$ vertices.*

Proof. The part (a) is clear from the definition. For claim (b), let us first show that, if Δ is banner, then every induced subcomplex Γ of Δ such that $\tilde{H}_{d-2}(\Gamma) \neq 0$ has at least $2d-2$ vertices by induction on d . If $d=3$, this is clear because Δ is flag.

Let $d > 3$. We may assume that no induced subcomplex of Δ has a nontrivial $(d-1)$ -dimensional cycle: indeed, such a subcomplex is forced to have dimension $d-1$, so it would be banner and we could replace Δ with it. Furthermore, we may assume that Γ is a minimal induced subcomplex with the property that $\tilde{H}_{d-2}(\Gamma) \neq 0$. Under such a minimality assumption, the link of any vertex of Γ admits a nontrivial homology cycle in dimension $d-3$. Take a vertex v of Γ . Since $\text{lk}_v \Gamma$ is an induced subcomplex of $\text{lk}_v \Delta$,

which is banner and admits a nontrivial $(d-2)$ -cycle, by induction $\text{lk}_v \Gamma$ has at least $2d-4$ vertices. Moreover, v cannot be a cone point of Γ because $\tilde{H}_{d-2}(\Gamma) \neq 0$, so $\#V(\Gamma) \geq 2d-2$.

The claim (b) now follows by induction on the banner number and claim (a). \square

Remark 10. While a flag simplicial complex (not necessarily of dimension $d-1$) supporting a nontrivial $(d-1)$ -cycle has at least $2d$ vertices, this is false for banner complexes. Take the boundary of a d -simplex, and join one facet with an external edge: the resulting complex is a $(d+1)$ -dimensional banner complex supporting a nontrivial $(d-1)$ -cycle, but with only $d+3$ vertices.

Lemma 11. *Let Δ be a pure $(d-1)$ -dimensional complex with nontrivial top-homology. If $b(\Delta) < d-2$, then $\mathbb{k}[\Delta]$ satisfies the property $\mathfrak{B}_{d-b(\Delta)-1}$.*

Proof. Notice that the regularity of $\mathbb{k}[\Delta]$ is d , since Δ has a nontrivial top-homology. If $b_{i,i+d-1}(\mathbb{k}[\Delta]) \neq 0$, by Hochster's formula there exists a subset $W \subseteq V(\Delta)$ of cardinality $i+d-1$ such that Δ_W supports a nontrivial $(d-2)$ -cycle. By part (b) of Lemma 9, thus:

$$i \geq d - b(\Delta) - 1. \quad \square$$

Theorem 12. *Let Δ be an $(d-1)$ -dimensional minimal cycle. Then the underlying graph of Δ is $(2d-b(\Delta)-2)$ -connected.*

Proof. If $b(\Delta) = d-2$, then it is easy to see that $\mathbb{k}[\Delta]$ satisfies \mathfrak{B}_1 . By Lemma 8 and Hochster's formula, $\mathbb{k}[\Delta]$ satisfies also the property \mathfrak{A} . Therefore, the result follows from Theorem 6.

If $b(\Delta) < d-2$, by Lemmata 8 (together with Hochster's formula) and 11, $\mathbb{k}[\Delta]$ satisfies the properties \mathfrak{A} and $\mathfrak{B}_{d-b(\Delta)-1}$. Therefore, the result follows from Theorem 6. \square

Corollary 13. *Let Δ be a flag (or more generally banner) $(d-1)$ -dimensional minimal cycle. Then the underlying graph of Δ is $(2d-2)$ -connected.*

Proof. If Δ is a banner complex, then $b(\Delta) = 0$. \square

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